

F -HARMONIC MAPS AS GLOBAL MAXIMA

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ABSTRACT. In this note, we show that some F -harmonic maps into spheres are global maxima of the variations of their energy functional on the conformal group of the sphere. Our result extends partially those obtained in [15] and [17] for harmonic and p -harmonic maps.

1. INTRODUCTION

Harmonic maps have been studied first by J. Eells and J.H.Sampson in the sixties and since then many articles have appeared (see [6], [12], [16], [19], [20], [24]) to cite a few of them. Extensions to the notions of p -harmonic, biharmonic, F -harmonic and f -harmonic maps were introduced and similar research has been carried out (see [1], [2], [3], [7], [15], [18], [21], [23]). Harmonic maps were applied to broad areas in sciences and engineering including the robot mechanics (see [5], [8], [9]).The concept of F - harmonic maps unifies the notions of harmonic maps, p -harmonic maps, minimal hypersurfaces. An important tool for studying stability of stability of F harmonic maps is the stress-energy tensor.

In this paper for a C^2 -function $F : [0, +\infty[\rightarrow [0, +\infty[$ such that $F'(t) > 0$ on $t \in]0, +\infty[$, we look for sufficient conditions which present F -harmonic maps into spheres as global maxima of the energy functional. Our result extends similar results obtained in [17] and [18] for harmonic and p -harmonic maps.

Let (M, g) and S^n be, respectively, a compact Riemannian manifold of dimension $m \geq 2$ and the unit n -dimensional Euclidean sphere with $n \geq 2$ endowed with the canonical metric *can* induced by the inner product of R^{n+1} .

For a C^1 - application $\phi : (M, g) \rightarrow (S^n, \text{can})$, we define the F -energy functional by

$$E_F(\phi) = \int_M F \left(\frac{|d\phi|^2}{2} \right) dv_g,$$

where $\frac{|d\phi|^2}{2}$ denotes the energy density given by

$$\frac{|d\phi|^2}{2} = \frac{1}{2} \sum_{i=1}^m |d\phi(e_i)|^2$$

and where $\{e_i\}$ is an orthonormal basis on the tangent space $T_x M$ and dv_g is the Riemannian measure associated to g on M .

Let $\phi^{-1}TS^n$ and $\Gamma(\phi^{-1}TS^n)$ be, respectively, the pullback vector fiber bundle of TS^n and the space of sections on $\phi^{-1}TS^n$. Denote by ∇^M , ∇^{S^n} and ∇ , respectively,

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the Levi-Civita connections on: TM , $T S^n$ and $\phi^{-1}TS^n$. Recall that ∇ is defined by

$$\nabla_X Y = \nabla_{\phi_* X}^S Y$$

where $X \in TM$ and $Y \in \Gamma(\phi^{-1}TS^n)$.

Let v be a vector field on S^n and denote by $(\gamma_t^v)_t$ the flow of diffeomorphisms induced by v on S^n i.e.

$$\gamma_0^v = id, \quad \frac{d}{dt} \gamma_{t=0}^v = v(\gamma_t^v).$$

Denote by $\phi_t = \gamma_t^v \circ \phi$ the flow generated by v along the map ϕ . The first variation formula of $E_F(\phi)$ is given by

$$\begin{aligned} \frac{d}{dt} E_F(\phi_t) |_{t=0} &= \int_M F' \left(\frac{|d\phi_t|^2}{2} \right) \langle \nabla_{\partial_t} d\phi_t, d\phi_t \rangle |_{t=0} dv_g \\ &= - \int_M \langle v, \tau_F(\phi) \rangle dv_g \end{aligned}$$

where $\tau_F(\phi) = trace_g \nabla \left(F' \left(\frac{|d\phi|^2}{2} \right) d\phi \right)$ is the F -tension.

Definition 1. ϕ is said F -harmonic if and only if $\tau_F(\phi) = 0$ i.e. ϕ is a critical point of the F -energy functional E_F .

Let $v \in \mathbb{R}^{n+1}$ and set $\bar{v}(y) = v - \langle v, y \rangle y$ for any $y \in S^n$. It is known that \bar{v} is a conformal vector field on S^n i.e. $(\gamma_t^v)^* can = \alpha_t^2 can$ where $(\gamma_t^v)_t$ denotes the flow induced by the vector field \bar{v} . The expression of α_t is given in [17] by

$$(1.1) \quad \alpha_t = \frac{|v|}{|v|cht + \phi_v sht}.$$

where $\phi_v(x) = \langle v, \phi(x) \rangle$ and $\langle \cdot, \cdot \rangle$ the inner product on the Euclidean space \mathbb{R}^{n+1} . Denote by $\mathcal{L}(\phi)$ the subspace of $\Gamma(\phi^{-1}TS^n)$ given by

$$\mathcal{L}(\phi) = \{ \bar{v} \circ \phi, v \in \mathbb{R}^{n+1} \}.$$

Obviously, if ϕ is not constant, $\mathcal{L}(\phi)$ is of dimension $n+1$.

2. F -HARMONIC MAPS AS GLOBAL MAXIMA

For any $\bar{v} \in \mathcal{L}(\phi)$, we denote by $(\gamma_t^v)_{t \in \mathbb{R}}$ the one parameter group of conformal diffeomorphisms on S^n induced by the vector \bar{v} . For a C^2 -function $F : [0, +\infty[\rightarrow [0, +\infty[$ such that $F'(t) > 0$ in $]0, +\infty[$.

Now we introduce the following tensor field

$$\begin{aligned} S_g^F(\phi) &= F' \left(\frac{|d\phi|^2}{2} \right) \frac{|d\phi|^2}{2} g \\ &\quad - \left[F' \left(\frac{|d\phi|^2}{2} \right) + \frac{|d\phi|^2}{2} F'' \left(\frac{|d\phi|^2}{2} \right) \right] \phi^* can. \end{aligned}$$

For $x \in M$, we set

$$S_g^{o,F}(\phi)(x) = \inf \{ S_{g,v}^F(\phi)(X, X), X \in T_x M \text{ such that } g(X, X) = 1 \}.$$

The tensor field $S_g^F(\phi)(x)$ will be said positive (resp. positive defined) at x if $S_g^{o,F}(\phi)(x) \geq 0$ (resp. $S_g^{o,F}(\phi)(x) > 0$). The tensor field $S_g^F(\phi)$ will be called

the F stress-energy tensor of ϕ . The tensor field $S_g^F(\phi)$ is different from the one defined by Ara given by $S_F(\phi) = F\left(\frac{|d\phi|^2}{2}\right)g - F'\left(\frac{|d\phi|^2}{2}\right)\phi^*can$, but $S_g^F(\phi)$ is more suitable for our case.

Example 1. For $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$, with $p = 2$ or $p \geq 4$, $S_g^p(\phi)$ is the stress-energy tensor introduced, respectively, by Eells and Lemaire for $p = 2$ [12] and modulo a multiplied positive constant by El Soufi for $p \geq 4$ [16], so we may call $S_g^F(\phi)$ the stress-energy tensor of ϕ .

Indeed if $F(t) = t$ then $F'(t) = 1$, $F''(t) = 0$ and

$$S_g^F(\phi) = \frac{|d\phi|^2}{2}g - \phi^*can.$$

In the case $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$, with $p \geq 4$, $F'(t) = (2t)^{\frac{p}{2}-1}$, $F''(t) = (p-2)(2t)^{\frac{p}{2}-2}$ and

$$S_g^F(\phi) = \frac{1}{2}|d\phi|^p g - \frac{p}{2}|d\phi|^{p-2}\phi^*can = \frac{p}{2}\left(\frac{1}{p}|d\phi|^p g - |d\phi|^{p-2}\phi^*can\right)$$

The function F is called *admissible* if it satisfies

$$B = \left(\frac{F''(\alpha_t^2 o\phi \cdot \frac{|d\phi|^2}{2})}{F'(\alpha_t^2 o\phi \cdot \frac{|d\phi|^2}{2})} \alpha_t^2 o\phi - \frac{F''\left(\frac{|d\phi|^2}{2}\right)}{F'\left(\frac{|d\phi|^2}{2}\right)} \right) \phi_v \geq 0$$

and the F stress-energy tensor $S_g^F(\phi)$ of ϕ fulfills

$$S_g^F(\gamma_t o\phi) \geq \theta(\alpha_t^2 o\phi) \cdot S_g^F(\phi)$$

where θ is a real positive function and γ_t is the one parameter group of conformal transformations induced by the vector field \bar{v} (defined above) on the euclidean sphere S^n and α_t is given by (1.1).

Example 2.

The function $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$ for $p = 2$ and $p \geq 4$ and $t \geq 0$ is admissible.

Indeed, for $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$ we have $B = 0$ and for any conformal diffeomorphism γ on the euclidean sphere, we have

$$S_g^F(\gamma o\phi) = \frac{1}{2}|d(\gamma_t o\phi)|^p g - \frac{p}{2}|d(\gamma_t o\phi)|^{p-2}\phi^*can$$

so if we let $|d(\gamma_t o\phi)|^2 = \alpha_t^2 o\phi \cdot |d\phi|^2$, we get

$$\begin{aligned} S_g^F(\gamma_t o\phi) &= \alpha_t^2 o\phi \cdot \left(\frac{1}{2}|d\phi|^p g - \frac{p}{2}|d\phi|^{p-2}\phi^*can \right) \\ &= \alpha_t^2 o\phi \cdot S_g^F(\phi). \end{aligned}$$

The $F(t) = 1 + at - e^{-t}$, for $t \in [0, +\infty[$ where $a = \max_{x \in M} \frac{|d\phi|^2}{2}$ is admissible provided that the conformal diffeomorphism on the euclidean sphere S^n is contracting that means that the function ϕ_v given in the expression of (1.1) is nonnegative and the stress-energy tensor $S_g(\phi) = \frac{|d\phi|^2}{2}g - \phi^*can$ of ϕ is positive.

Indeed, we have

$$B = \left(-\alpha_t^2 o\phi \frac{e^{-\alpha_t^2 o\phi \cdot \frac{|d\phi|^2}{2}}}{a + e^{-\alpha_t^2 o\phi \cdot \frac{|d\phi|^2}{2}}} + \frac{e^{-\frac{|d\phi|^2}{2}}}{a + e^{-\frac{|d\phi|^2}{2}}} \right) \phi_v$$

Putting $u = \alpha_t^2 o\phi \in]0, 1]$, we consider the function $\varphi(u) = -u \frac{e^{-u \frac{|d\phi|^2}{2}}}{a + e^{-u \frac{|d\phi|^2}{2}}} + \frac{e^{-\frac{|d\phi|^2}{2}}}{a + e^{-\frac{|d\phi|^2}{2}}}$, we get

$$\varphi'(u) = \left(\frac{|d\phi|^2}{2} u - a - e^{-|d\phi|^2 u} \right) \frac{e^{-\frac{|d\phi|^2}{2}}}{\left(a + e^{-\frac{|d\phi|^2}{2}} u \right)^2}$$

and it is obvious that $\varphi'(u) \leq 0$, hence φ is a decreasing function on $]0, 1]$ i.e. $\varphi(u) \geq \varphi(1) = 0$. Consequently $B \geq 0$.

Now

$$\begin{aligned} S_g^F(\gamma_t o\phi)(X; X) &= \left(a + e^{-\frac{|d(\gamma_t o\phi)|^2}{2}} \right) \frac{|d(\gamma_t o\phi)|^2}{2} g(X, X) \\ &- \left[\left(a + e^{-\frac{|d(\gamma_t o\phi)|^2}{2}} \right) - \frac{|d(\gamma_t o\phi)|^2}{2} e^{-\frac{|d(\gamma_t o\phi)|^2}{2}} \right] (\gamma_t o\phi)^* \text{can}(X, X) \\ &= \alpha_t^2 o\phi \left(a + e^{-\alpha_t^2 o\phi \frac{|d\phi|^2}{2}} \right) \frac{|d\phi|^2}{2} g(X, X) \\ &- \alpha_t^2 o\phi \left[\left(a + e^{-\alpha_t^2 o\phi \frac{|d\phi|^2}{2}} \right) - \alpha_t^2 o\phi \frac{|d\phi|^2}{2} e^{-\alpha_t^2 o\phi \frac{|d\phi|^2}{2}} \right] \phi^* \text{can}(X, X) \\ &= \alpha_t^2 o\phi \left(a + e^{-\frac{|d(\gamma_t o\phi)|^2}{2}} \right) \left[\frac{1}{2} |d\phi|^2 g(X, X) - \phi^* \text{can}(X, X) \right] \\ &\quad + \alpha_t^2 o\phi \frac{|d\phi|^2}{2} e^{-\alpha_t^2 o\phi \frac{|d\phi|^2}{2}} \phi^* \text{can}(X, X). \end{aligned}$$

An other example is the following function $F(t) = (1 + 2t)^\alpha$ where $0 < \alpha < 1$, the F -energy is the α -energy of Sacks-Uhlenbeck (see [22]). In fact

$$\begin{aligned} B &= (\alpha - 1) \left(\frac{1}{1 + \alpha_t^2 o\phi \cdot |d\phi|^2} \alpha_t^2 o\phi - \frac{1}{1 + |d\phi|^2} \right) \phi_v \\ &= \frac{(\alpha - 1) (\alpha_t^2 o\phi - 1)}{(1 + \alpha_t^2 o\phi \cdot |d\phi|^2) (1 + |d\phi|^2)} \phi_v \geq 0 \end{aligned}$$

provided that $\phi_v \geq 0$.

And for vector field X on M , we have

$$\begin{aligned} S_g^F(\gamma_t o\phi)(X; X) &= 2\alpha \left(1 + \alpha_t^2 o\phi |d\phi|^2 \right)^{\alpha-1} \alpha_t^2 o\phi \frac{|d\phi|^2}{2} g(X, X) - \\ &\left[2\alpha \left(1 + \alpha_t^2 o\phi |d\phi|^2 \right)^{\alpha-1} + 2\alpha (\alpha - 1) \left(1 + \alpha_t^2 o\phi |d\phi|^2 \right)^{\alpha-2} \alpha_t^2 o\phi \cdot |d\phi|^2 \right] \alpha_t^2 o\phi \cdot \phi^* \text{can}(X, X) \\ &= \left[2\alpha \left(1 + \alpha_t^2 o\phi |d\phi|^2 \right)^{\alpha-1} \alpha_t^2 o\phi \left(\frac{|d\phi|^2}{2} g(X, X) - \phi^* \text{can}(X, X) \right) + \right. \\ &\quad \left. 2\alpha (1 - \alpha) \left(1 + \alpha_t^2 o\phi |d\phi|^2 \right)^{\alpha-2} \alpha_t^2 o\phi |d\phi|^2 \cdot \phi^* \text{can}(X, X) \right] \end{aligned}$$

and taking account of the positivity of the stress-energy tensor of $S_g(\phi) = \frac{1}{2}|d\phi|^2 g - \phi^* \text{can}$ and the fact that $\phi_v \geq 0$, we infer that

$$S_g^F(\gamma_t \circ \phi)(X, X) \geq \alpha_t^4 \circ \phi \cdot S_g^F \phi(X, X).$$

Remark 1. $\phi_v \geq 0$ occurs for example if $\phi(M)$ is included in the positive half-sphere $S^{n+} = \{x \in S^n : \langle x, v \rangle \geq 0\}$.

In this section we state the following result

Theorem 1. Let $F : [0, +\infty[\rightarrow [0, +\infty[$ be an admissible function and ϕ be an F -harmonic map from a compact m -Riemannian manifold (M, g) ($m \geq 2$) into the Euclidean sphere S^n ($n \geq 2$). Suppose that the F stress-energy tensor $S_g^F(\phi)$ is positive (resp. positively defined). Then for any conformal diffeomorphism γ on S^n , $E_F(\gamma \circ \phi) \leq E_F(\phi)$ (resp. $E_F(\gamma \circ \phi) < E_F(\phi)$).

Remark 2. In case $F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$, $p = 2$ or $p \geq 4$ the condition $\phi_v \geq 0$ is not needed since $B = 0$, so our result recover the ones by El-Soufi in [16] and [18].

To prove Theorem 1, we need the following lemmas

Lemma 1. Let $\phi : (M, g) \rightarrow (N, h)$ be a smooth map and γ be a conformal diffeomorphism on N , then the F -tension of the map $\gamma \circ \phi$, is given by

$$\begin{aligned} \tau_F(\gamma \circ \phi) &= 2\alpha^{-1} \circ \phi \cdot F'(\alpha^2 \circ \phi \frac{|d\phi|^2}{2}) d\gamma \left(d\phi_v - \frac{|d\phi|^2}{2} \nabla \alpha \circ \phi \right) \\ &\quad + f d\gamma(\tau_F(\phi)) + d\gamma \left(F' \left(\frac{|d\phi|^2}{2} \right) d\phi(\nabla f) \right). \end{aligned}$$

where $f = \frac{F'(\alpha^2 \circ \phi \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})}$ and $\gamma^* \text{can} = \alpha^2 \text{can}$.

Proof. We follow closely the proof in [18]

$$\begin{aligned} \tau_F(\gamma \circ \phi) &= \text{trace}_g \nabla \left(F' \left(\frac{|d(\gamma \circ \phi)|^2}{2} \right) d(\gamma \circ \phi) \right) = F' \left(\frac{|d(\gamma \circ \phi)|^2}{2} \right) \text{trace}(\nabla d(\gamma \circ \phi)) \\ &\quad + d(\gamma \circ \phi) \left(\nabla F' \left(\frac{|d(\gamma \circ \phi)|^2}{2} \right) \right) \end{aligned}$$

where $\nabla F' \left(\frac{|d(\gamma \circ \phi)|^2}{2} \right)$ is the gradient of $F' \left(\frac{|d(\gamma \circ \phi)|^2}{2} \right)$ in M .

Since γ is a conformal diffeomorphism on S^n , we have

$$\begin{aligned} \tau_F(\gamma \circ \phi) &= F' \left(\alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) \tau(\gamma \circ \phi) + d(\gamma \circ \phi) \left(\nabla F' \left(\alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) \right) \\ &= F' \left(\alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) (\text{trace}_g \nabla^\gamma d\gamma(d\phi, d\phi) + d\gamma \cdot \tau(\phi)) + d(\gamma \circ \phi) \left(\nabla F' \left(\alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) \right) \\ &= F' \left(\alpha^2 \circ \phi \frac{|d\phi|^2}{2} \right) \left(\text{trace}_g \nabla^\gamma d\gamma(d\phi, d\phi) + \frac{1}{F' \left(\frac{|d\phi|^2}{2} \right)} d\gamma(\tau_F(\phi)) \right) \end{aligned}$$

$$-\frac{F'(\alpha^2 o\phi \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})} d(\gamma o\phi) \left(\nabla F' \left(\frac{|d\phi|^2}{2} \right) \right) + d(\gamma o\phi) \left(\nabla F' \left(\alpha^2 o\phi \frac{|d\phi|^2}{2} \right) \right).$$

Putting $f = \frac{F'(\alpha^2 o\phi \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})}$ we get

$$\begin{aligned} \tau_F(\gamma o\phi) &= F'(\alpha^2 o\phi \frac{|d\phi|^2}{2}) \text{trace}_g \nabla^\gamma d\gamma(d\phi, d\phi) + f d\gamma(\tau_F(\phi)) \\ &\quad + F' \left(\frac{|d\phi|^2}{2} \right) d(\gamma o\phi)(\nabla f). \end{aligned}$$

Now since $\gamma : (N, \gamma^* can) \rightarrow (N, can)$ is an isometry then, if $\tilde{\nabla}$ denotes the connection corresponding to $\gamma^* can$, we have

$$\nabla^\gamma d\gamma(X, Y) = d\gamma(\tilde{\nabla}_X Y - \nabla_X Y)$$

and since (see [18])

$$\tilde{\nabla}_X Y - \nabla_X Y = \alpha^{-1} (\langle X, \nabla \alpha \rangle Y + \langle Y, \nabla \alpha \rangle X - \langle X, Y \rangle \nabla \alpha)$$

we obtain

$$\text{trace}_g \nabla^\gamma d\gamma(d\phi, d\phi) = 2\alpha^{-1} o\phi \cdot d\gamma \left(d\phi(\nabla \alpha o\phi) - \frac{|d\phi|^2}{2} \nabla \alpha o\phi \right).$$

Finally we infer that

$$\begin{aligned} \tau_F(\gamma o\phi) &= 2\alpha^{-1} o\phi \cdot F'(\alpha^2 o\phi \frac{|d\phi|^2}{2}) d\gamma \left(d\phi(\nabla \alpha o\phi) - \frac{|d\phi|^2}{2} \nabla \alpha o\phi \right) \\ &\quad + f d\gamma(\tau_F(\phi)) + F' \left(\frac{|d\phi|^2}{2} \right) d\gamma o\phi(\nabla f). \end{aligned}$$

□

Lemma 2. *Let ϕ be an F -harmonic map from an m -dimensional Riemannian manifold (M, g) ($m \geq 2$) into the Euclidean unit sphere (S^n, can) ($n \geq 2$).*

Then for any $v \in R^{n+1} - \{0\}$ and any $t_o \in R$ we have

$$\begin{aligned} \frac{d}{dt} E_F(\gamma_t^v o\phi)_{t=t_o} &= \\ -2 \frac{sh t_o}{|v|} \int_M \alpha_{t_o}^3 o\phi \cdot F' \left(\alpha_{t_o}^2 o\phi \cdot \frac{|d\phi|^2}{2} \right) (|d\phi|^2 |\bar{v} o\phi|^2 - |d\phi_v|^2) dv_g \\ &\quad - \int_M \alpha_{t_o}^2 o\phi \cdot F' \left(\frac{|d\phi|^2}{2} \right) \langle d\phi(\nabla f_{t_o}), \bar{v} o\phi \rangle dv_g \end{aligned}$$

where $f_{t_o} = \frac{F'(\alpha_{t_o}^2 o\phi \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})}$, $\gamma^* can = \alpha_{t_o}^2 can$

and

$$\alpha_{t_o} = \frac{|v|}{\phi_v sh t_o + |v| ch t_o}.$$

Proof. Recall that the first variation formula of the F -energy is given by

$$\frac{d}{dt} E_F(\gamma_t^v o\phi)_{t=t_o} = - \int_M \langle \tau_F(\gamma_{t_o}^v o\phi), \bar{v} o(\gamma_{t_o}^v o\phi) \rangle dv_g.$$

By Lemma 1 and the fact that ϕ is F -harmonic we get

$$\begin{aligned} \frac{d}{dt} E_F(\gamma_t^v o\phi)_{t=t_o} = \\ - \int_M 2\alpha_{t_o} o\phi.F' \left(\alpha_{t_o}^2 o\phi. \frac{|d\phi|^2}{2} \right) \left\langle (\nabla \alpha_{t_o} o\phi)^T - \frac{|d\phi|^2}{2} \nabla \alpha_{t_o} o\phi, \bar{v} o\phi \right\rangle dv_g \\ - \int_M F' \left(\frac{|d\phi|^2}{2} \right) \langle d\phi(\nabla f_{t_o}), \bar{v} o\phi \rangle dv_g \end{aligned}$$

and since (see [18])

$$(2.1) \quad \nabla \alpha_{t_o}^v = - \frac{(\alpha_{t_o}^v)^2}{|v|} sht_o \bar{v}$$

we have

$$(2.2) \quad \langle \nabla \alpha_{t_o} o\phi, \bar{v} o\phi \rangle = - \frac{(\alpha_{t_o}^v)^2}{|v|} sht_o |\bar{v} o\phi|^2.$$

Now let (e_1, \dots, e_m) be an orthogonal basis on M

$$\begin{aligned} \langle (\nabla \alpha_{t_o} o\phi)^T, \bar{v} o\phi \rangle &= \sum_{i=1}^m \langle \nabla \alpha_{t_o} o\phi, d\phi(e_i) \rangle \langle \bar{v} o\phi, d\phi(e_i) \rangle \\ &= - \frac{sht_o}{|v|} (\alpha_{t_o} o\phi)^2 \sum_{i=1}^m \langle \bar{v} o\phi, d\phi(e_i) \rangle^2 \\ &= - \frac{sht_o}{|v|} (\alpha_{t_o} o\phi)^2 |d\phi_v|^2. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt} E_F(\gamma_t^v o\phi)_{t=t_o} = \\ - 2 \frac{sht_o}{|v|} \int_M \alpha_{t_o}^3 o\phi.F' \left(\alpha_{t_o}^2 o\phi. \frac{|d\phi|^2}{2} \right) \left(\frac{|d\phi|^2}{2} |\bar{v} o\phi|^2 - |d\phi_v|^2 \right) dv_g \\ - \int_M \alpha_{t_o}^2 o\phi.F' \left(\frac{|d\phi|^2}{2} \right) \langle d\phi(\nabla f_{t_o}), \bar{v} o\phi \rangle dv_g. \end{aligned}$$

□

We set

$$g(t) = \int_M \alpha_t^2 o\phi.F' \left(\frac{|d\phi|^2}{2} \right) \langle d\phi(\nabla f_t), \bar{v} o\phi \rangle dv_g.$$

Lemma 3.

$$g(t) = \int_M \alpha_t^3 o\phi . F''(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2}) |d\phi|^2 \langle d\phi (\nabla (\alpha_{t_o} o\phi)), \bar{v} o\phi \rangle dv_g \\ + \int_M \alpha_t^2 o\phi . \left(F''(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2}) \alpha_t^2 o\phi - \frac{F'(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})} F''\left(\frac{|d\phi|^2}{2}\right) \right) \frac{\phi_v}{|v|} |d\phi|^2 dv_g.$$

Proof. First, we compute ∇f_t

$$\nabla f_t = \frac{F''(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})} \left(\alpha_t o\phi . |d\phi|^2 \nabla (\alpha_t o\phi) + \alpha_t^2 o\phi . \langle \nabla d\phi, d\phi \rangle \right) \\ - \frac{F'(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})^2} F''\left(\frac{|d\phi|^2}{2}\right) \langle \nabla d\phi, d\phi \rangle \\ = \frac{F''(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})} \alpha_t o\phi |d\phi|^2 \nabla (\alpha_t o\phi) \\ + \left(\frac{F''(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})} \alpha_t^2 o\phi - \frac{F'(\alpha_t^2 o\phi . \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})^2} F''\left(\frac{|d\phi|^2}{2}\right) \right) \langle \nabla d\phi, d\phi \rangle.$$

Then

$$g(t) = \int_M \alpha_{t_o}^2 o\phi . F'\left(\frac{|d\phi|^2}{2}\right) \langle d\phi (\nabla f_{t_o}), \bar{v} o\phi \rangle dv_g \\ = \int_M \alpha_{t_o}^3 o\phi . F''(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2}) |d\phi|^2 \langle d\phi (\nabla (\alpha_{t_o} o\phi)), \bar{v} o\phi \rangle dv_g \\ (2.3) \quad + \int_M \alpha_{t_o}^2 o\phi . \left(F''(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2}) \alpha_{t_o}^2 o\phi - \frac{F'(\alpha_{t_o}^2 o\phi . \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})} F''\left(\frac{|d\phi|^2}{2}\right) \right) \\ \times \langle d\phi (\nabla |d\phi|^2), \bar{v} o\phi \rangle dv_g.$$

Let $\{e_1, \dots, e_m\}$ be a basis of $T_x M$ which diagonalizes $\phi^* can$, we have

$$\left\langle d\phi \left(\nabla \frac{|d\phi|^2}{2} \right), \bar{v} o\phi \right\rangle = \langle \nabla_{e_i} d\phi, d\phi \rangle \langle \bar{v} o\phi, d\phi(e_j) \rangle \langle d\phi(e_i), d\phi(e_j) \rangle \\ = \langle \nabla_{e_i} d\phi, d\phi \rangle \langle \bar{v} o\phi, d\phi(e_j) \rangle \phi^* can(e_i, e_j) \\ = \langle \nabla_{\bar{v} o\phi} d\phi(e_j), d\phi(e_j) \rangle \\ = \langle \nabla_{d\phi(e_j)} \bar{v} o\phi, d\phi(e_j) \rangle + \langle [\bar{v} o\phi, d\phi(e_j)], d\phi(e_j) \rangle.$$

Likewise we get

$$\langle [\bar{v} o\phi, d\phi(e_j)], d\phi(e_j) \rangle = -\frac{1}{2} \frac{d}{dt} \Big|_{t=0} \gamma_t^* |d\phi(e_j)|^2 \\ = -\frac{1}{2} \frac{d}{dt} \Big|_{t=0} \alpha_t^2 |d\phi(e_j)|^2$$

and taking account of (1.1) we obtain that

$$\langle [\bar{v}o\phi, d\phi(e_j)], d\phi(e_j) \rangle = \frac{\phi_v}{|v|} |d\phi(e_j)|^2$$

so we infer that

$$\left\langle d\phi \left(\nabla \frac{|d\phi|^2}{2} \right), \bar{v}o\phi \right\rangle = \frac{\phi_v}{|v|} |d\phi|^2 + \langle \nabla_{e_j} \bar{v}o\phi, d\phi(e_j) \rangle$$

and

$$\begin{aligned} \langle \nabla_{e_j} \bar{v}o\phi, d\phi(e_j) \rangle &= \nabla_{e_j} \langle \bar{v}o\phi, d\phi(e_j) \rangle - \langle \bar{v}o\phi, \nabla_{e_j} d\phi(e_j) \rangle \\ &= \nabla_{e_j} \langle v, d\phi(e_j) \rangle - \langle \bar{v}o\phi, \nabla_{e_j} d\phi(e_j) \rangle \\ &= \langle v, \nabla_{e_j} d\phi(e_j) \rangle - \langle \bar{v}o\phi, \nabla_{e_j} d\phi(e_j) \rangle \\ &= \langle v - \bar{v}o\phi, \nabla_{e_j} d\phi(e_j) \rangle = 0. \end{aligned}$$

Hence

$$(2.4) \quad \left\langle d\phi \left(\nabla \frac{|d\phi|^2}{2} \right), \bar{v}o\phi \right\rangle = \frac{\phi_v}{|v|} |d\phi|^2.$$

□

Now set

$$\begin{aligned} \varphi(t_o) &= 2 \frac{sht_o}{|v|} \int_M \alpha_{t_o}^3 o\phi \cdot F' \left(\alpha_{t_o}^2 o\phi \cdot \frac{|d\phi|^2}{2} \right) \left(-\frac{|d\phi|^2}{2} |\bar{v}o\phi|^2 + |d\phi_v|^2 \right) dv_g \\ &\quad - \int_M \alpha_{t_o}^3 o\phi \cdot F'' \left(\alpha_{t_o}^2 o\phi \cdot \frac{|d\phi|^2}{2} \right) \frac{|d\phi|^2}{2} \langle d\phi(\nabla(\alpha_{t_o} o\phi)), \bar{v}o\phi \rangle dv_g \end{aligned}$$

and since by (2.1) we have

$$\langle d\phi(\nabla(\alpha_{t_o} o\phi)), \bar{v}o\phi \rangle = -\frac{sht_o}{|v|} \alpha_{t_o}^2 o\phi |d\phi_v|^2$$

we get

$$\begin{aligned} (2.5) \quad \varphi(t_o) &= 2 \frac{sht_o}{|v|} \int_M \alpha_{t_o}^3 o\phi \cdot \left[\left(F' \left(\alpha_{t_o}^2 o\phi \cdot \frac{|d\phi|^2}{2} \right) + \alpha_{t_o}^2 o\phi \cdot \frac{|d\phi|^2}{2} F'' \left(\alpha_{t_o}^2 o\phi \cdot \frac{|d\phi|^2}{2} \right) \right) |d\phi_v|^2 \right. \\ &\quad \left. - F' \left(\alpha_{t_o}^2 o\phi \cdot \frac{|d\phi|^2}{2} \right) \frac{|d\phi|^2}{2} |\bar{v}o\phi|^2 \right] dv_g. \end{aligned}$$

or

$$\varphi(t_o) = -2 \frac{sht_o}{|v|} \int_M \alpha_{t_o}^3 o\phi \cdot S_g^F(\gamma_t^v o\phi) dv_g.$$

Proof. (of Theorem 1) Recall (see [13]) that for any conformal diffeomorphism γ of the unit sphere S^n there exist an isometry $r \in O(n+1)$, a real number $t \geq 0$ and a vector $v \in R^{n+1} - \{0\}$ such that $\gamma = r \circ \gamma_t^v$, so it suffices to consider γ_t^v with $t \geq 0$ and $v \in R^{n+1} - \{0\}$.

On the other hand

$$\frac{d}{dt} E_F(\gamma_t^v o\phi) = \varphi(t) + \chi(t)$$

where

$$\chi(t) = - \int_M \alpha_t^2 o\phi. \left(F''(\alpha_t^2 o\phi. \frac{|d\phi|^2}{2}) \alpha_t^2 o\phi - \frac{F'(\alpha_t^2 o\phi. \frac{|d\phi|^2}{2})}{F'(\frac{|d\phi|^2}{2})} F''\left(\frac{|d\phi|^2}{2}\right) \right) \frac{\phi_v}{|v|} |d\phi|^2 dv_g$$

and $\varphi(t)$ is given by (2.5). Now, since the function F is admissible we infer that $\chi(t) \leq 0$. Since the F energy-stress tensor $S_g^F(\phi)$ of ϕ is positive (resp. positive defined) by assumption and

$$\geq S_g^F(\phi)$$

so the tensor field $S_g^F(\gamma_t o\phi)$ is positive (resp. positive defined). Consequently $\varphi(t) \leq 0$ (resp. $\varphi(t) < 0$) for any $t \geq 0$ and the proof of Theorem 1 is complete. \square

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